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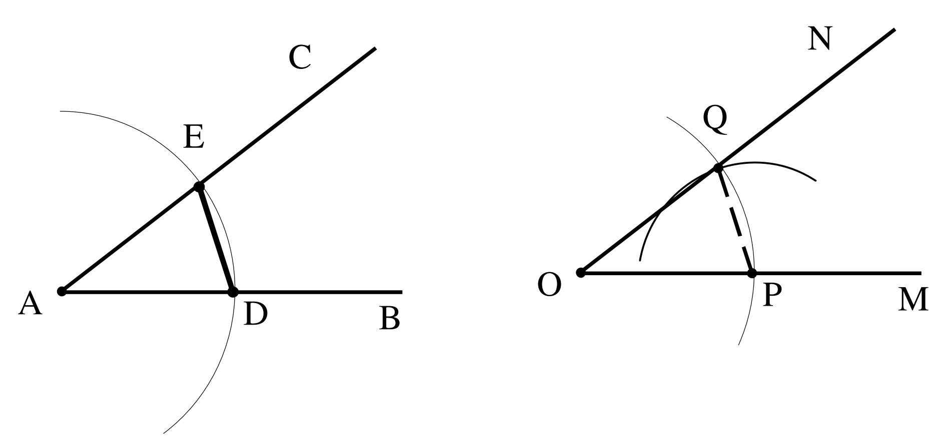
## *Sample 3*

### 5.2 Basic construction problems.

a) <u>To construct an angle congruent to a given angle, with the vertex at a given point and one side lying in a given line.</u>

∠ CAB is given (Figure 5.2.1a); also, point O and ray OM emanating form O (Figure 5.2.1b) are given. We have to construct an angle congruent to ∠CAB with the vertex at O and one side coinciding with OM.




(a) Figure 5.2.1 (b)

(i) If the required angle NOM is constructed, then any triangle, e.g., QOP, with two sides OQ and OP including ∠ NOM, will be congruent to the Δ EAD with AE = OQ and AD = OP, since these pairs of respectively equal sides include equal angles, NOM and CAB.

Thus, we can construct the required angle by constructing a triangle with its base lying on OM with one of its extremities at O, congruent to a triangle containing CAB as an angle.

(i) Now let us construct. With centre A and an arbitrary radius, describe an arc cutting AB at D and AC at E. With centre O and the same radius, describe an arc cutting OM at some point, P. With centre P and radius equal DE, describe an arc cutting the former arc at Q. Draw a ray, ON, emanating from O and passing through Q. NOM is the sought angle.

(ii) Let us prove it. Δ QOP = Δ EAD by *SSS* condition, then

∠ QOP = ∠ EAD, and OP lies along OM. Then ∠ QOP (or ∠ NOM) is the required one.

(iii) How many solutions does the problem have? Apparently, the arcs described from O and P will intersect at two points, Q and $Q_1$ (Fig. 5.2.2.), and ∠ $Q_1OP$ also satisfies the condition of the problem.

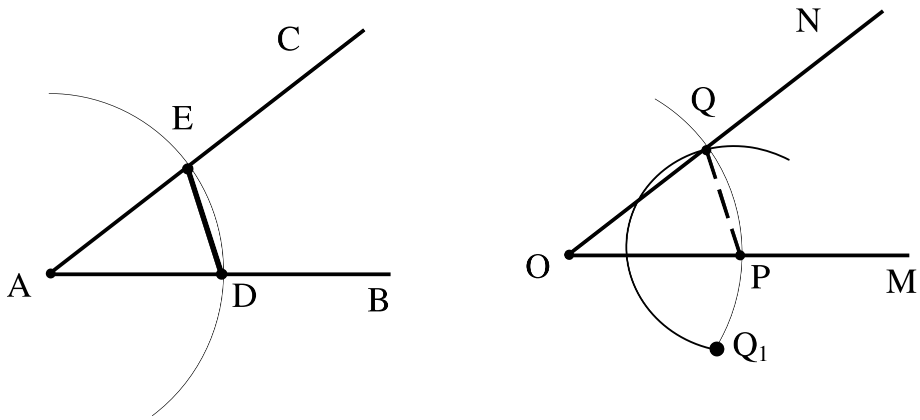



Figure 5.2.2

Thus, the problem has two solutions: there is an angle congruent to ∠ CAB on the either side of OM.

b) <u>To construct a triangle having given the three sides (*SSS*).</u>

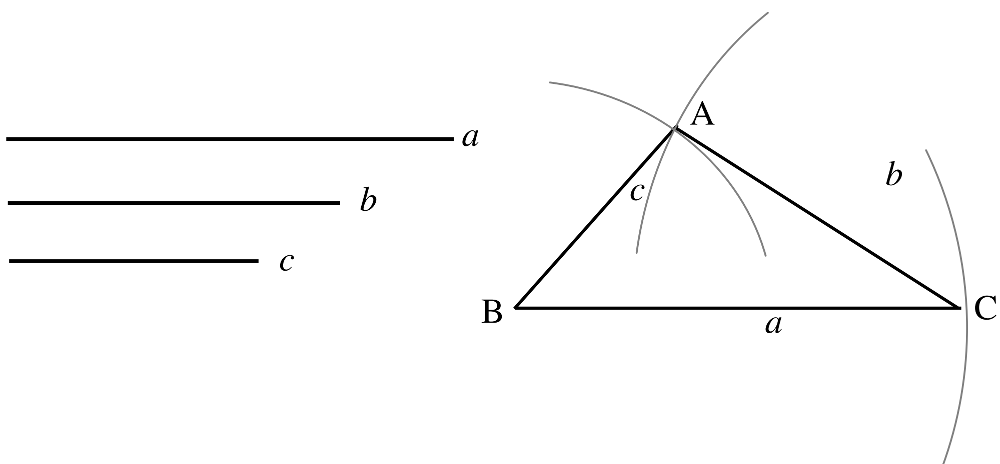



Figure 5.2.3

Three segments, - *a, b, c,* are given. It should be noticed that for each of them the triangle inequality is valid. Draw a ray emanating from some point B and describe about B an arc of a circle of radius *a*. The arc will intersect the ray at some point, C. From B and C describe arcs of radii *c* and *b*, respectively. They will intersect at some point, A.

Now BC = *a*, AC = *b*, AB = *c*; thus, we have constructed the required triangle. Any other Δ with the same sides will be congruent to Δ ABC.

c) <u>To construct a triangle having given two sides and the included angle (*SAS*).</u>

To construct an angle congruent to the given one and from the vertex to describe arcs with radii equal to the two given sides, making each arc to intersect one arm of the angle. Join the points of the intersections by a straight segment. The obtained $\Delta$ is uniquely defined by the *SAS* condition.

d) To construct a triangle having given two angles and the included side (*ASA*).

(To be solved by students as an exercise).

e) To construct a triangle having given two sides and an angle opposite to one of them (*SSA*).

Draw a ray, AM, emanating from A, and construct (as in problem [a]) another ray, AN, forming angle NAM congruent to the given angle $\alpha$.

From A describe an arc of radius equal to one of the given sides, *b*. The arc intersects AN at some point, C. From C

describe a circle with radius equal to the other given side, *a*. This circle may intersect AM at two points, one point, or none points depending on whether *a* is greater, equal or less than the perpendicular, *h*, dropped from C onto AM (Figure 5.2.4.).

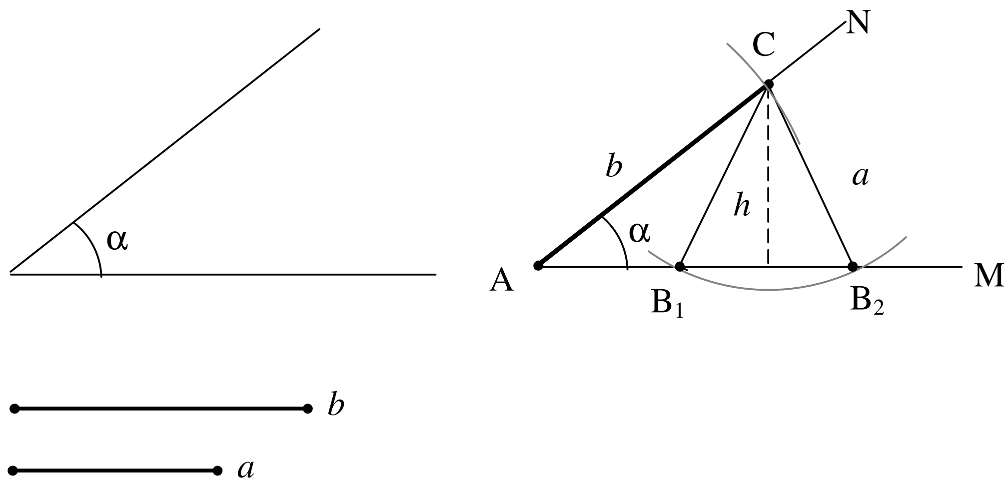



Figure 5.2.4

If $b>a>h$, both $\Delta AB_1C$ and $\Delta AB_2C$ satisfy the given condition, thus, the problem has two different solutions. This double solution is known as *the Ambiguous Case*.

If $a > b$ and $a > h$, there is a single solution (Figure 5.2.5.).

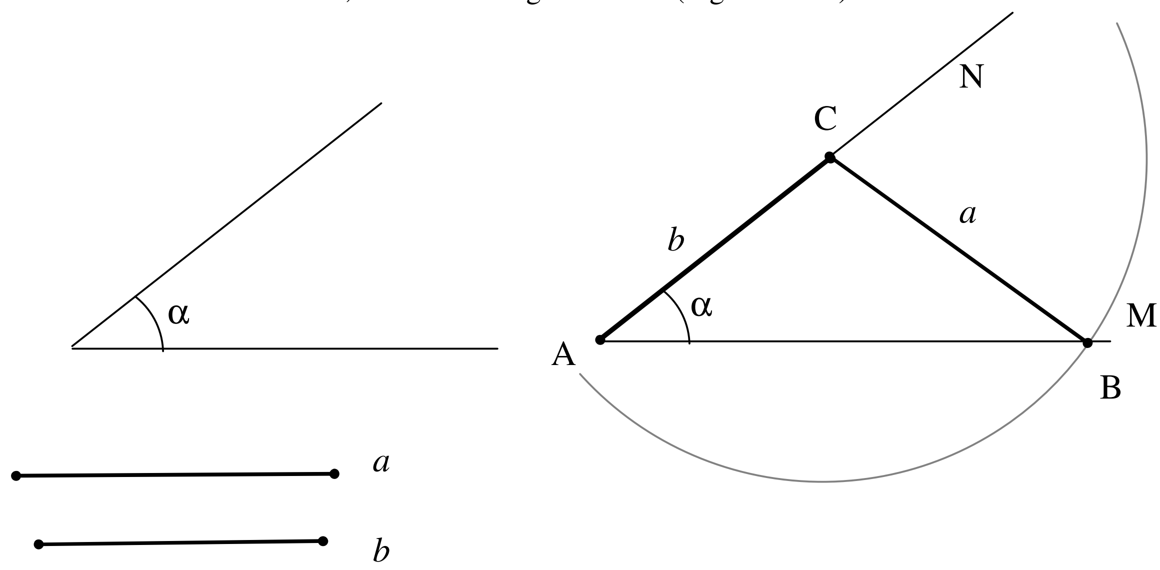



Figure 5.2.5

If $a = h$, the arc touches AM at one point only (this point is the foot of the perpendicular dropped from C onto AM). Then, there is a single solution in this case, and this solution is a right triangle. The latter is not surprising if we recall Th. 4.9.1 saying that given the hypotenuse (in our case, $b$) and a leg (in our case, $a$), the right triangle is determined uniquely.