Sample 2

2.3 Direct, converse, inverse, and contrapositive theorems.

As we have already mentioned, each theorem can be formulated in the *If… then…* form that is also called an *implication* or a *conditional statement*.

Denoting the hypothesis (condition) of a theorem by $\boldsymbol{H}$ and the conclusion by $\boldsymbol{C}$ (hereafter we shall use capital Latin letters to denote statements), we can symbolically represent the *If* $\boldsymbol{H}$ *, then* $\boldsymbol{C}$ theorem as $\boldsymbol{H} \Rightarrow \boldsymbol{C}$. It is read as $\boldsymbol{C}$ *follows from* $\boldsymbol{H}$ or $\boldsymbol{H}$ *implies* $\boldsymbol{C}$. Mathematicians also say: $\boldsymbol{H}$ *is sufficient for* $\boldsymbol{C}$ or $\boldsymbol{C}$ *is necessary upon* $\boldsymbol{H}$.

The *if* clause, ***H*,** is called the *antecedent* and the *then* clause, ***C*,** the *consequent*.

The theorem formed by interchanging the hypothesis (antecedent) and the conclusion (consequent) of a given *direct* theorem, is called the *converse* of the given theorem. For example, Th.2.2.2,

"if chords are congruent $\Rightarrow$ the respective arcs are congruent" is the converse of Th.2.2.1,

"if the arcs are congruent $\Rightarrow$ the respective chords are congruent".

It is not always the case that the converse of a true statement (theorem) is true. For example, let us consider the following conditional statement:

I live in Edmonton $\Rightarrow$ *I live in Alberta.*

The converse statement would be:

I live in Alberta $\Rightarrow$ *I live in Edmonton,*

that is evidently wrong (also, see the picture at the end of the section).

If the converse of a true theorem is true, as for instance, in the case of Th. 2.2.2 and 2.2.1, one says that the hypothesis and the conclusion of the theorem are logically *equivalent*:

$\boldsymbol{H} \Leftrightarrow \boldsymbol{C}$, or $\boldsymbol{H}$ iff $\boldsymbol{C}$

(the latter is a short notation for: $\boldsymbol{H}$ *is true if and only if* $\boldsymbol{C}$ *is true).*

The *inverse* of a given theorem is formed by negating both the hypothesis (antecedent) and the conclusion (consequent): $\overline{\boldsymbol{H}}$ Þ $\overline{\boldsymbol{C}}$**,** that means: if the hypothesis is false, then the conclusion is false.

For the conditional statement: *I live in Edmonton* ⇒ *I live in Alberta,* the inverse would be: *I do not live in Edmonton* ⇒ *I do not live in Alberta,* that is, of course, false. This example shows that in general **the inverse of a true statement is not true.**

It should be noticed here that when mathematicians say “is not true”, they mean “is not true in general” or “is not necessarily true”. However, “is not true” in mathematical sense does not exclude the option “sometimes is true”. Thus, some inverses of valid statements are true. For instance, the inverse of Th. 2.2.1 is *unequal arcs in the same circle are subtended by unequal chords*, that is, of course, true (one can prove). Another example: a mathematician would say that *an integer is not even* (in general), however one would agree that some integers are even.

One can also observe another *important* feature of deductive (logical) systems: one can disprove a statement by one counterexample (that is how we have shown that the inverse [also the converse] of a valid statement is not valid). Still one cannot prove a statement by adducing examples in favour of this statement. E.g., one can show many people who don’t live in Edmonton and don’t live in Alberta, and yet that does not mean that *I do not live in Edmonton* implies *I do not live in Alberta*.

It is easy to see that for a given conditional statement (no matter whether it is true or false) its **converse and inverse are logically equivalent**, i.e. if the converse is true, then the inverse is true and vice versa. Really, let us suppose (using our standard notation) that for some proposition ***H*** Þ ***C*** (that may be true or false), its converse is true: ***C*** Þ ***H***. Then, if ***H*** is not true, ***C*** cannot be true, because ***C*** necessarily implies ***H*** (***H*** is necessary if ***C*** is true). Translating the latter into formal notation, we obtain: ***H*** is not true ⇒ ***C*** is not true, or $\overline{\boldsymbol{H}}$ Þ $\overline{\boldsymbol{C}}$**,** the inverse statement., Q. E. D. Similarly, one can prove that the inverse implies the converse, and thus, the inverse and the converse are equivalent.

Finally, the *contrapositive* of a given conditional statement $\boldsymbol{H} \Rightarrow \boldsymbol{C}$ is $\overline{\boldsymbol{C}} \Rightarrow \overline{\boldsymbol{H}}$ formed by interchanging the hypothesis (antecedent) and the conclusion (consequent) and negating both of them.

For our example *I live in Edmonton* $\Rightarrow$ *I live in Alberta,*

the contrapositive will be : *I do not live in Alberta* $\Rightarrow$ *I do not live in Edmonton.*

One can easily show that **a direct statement is logically equivalent to its contrapositive**.

This property of conditional statements is called *contrapositive reasoning.* This is the kind of reasoning we use when proving conditional statements *by contradiction* (RAA *method*). Let us see how it works. Suppose, we have to show that if $\boldsymbol{H}$ is true, then $\boldsymbol{C}$ is true: $\boldsymbol{H} \Rightarrow \boldsymbol{C}$. We suggest that $\overline{\boldsymbol{C}}$ is true, which is equivalent to $\boldsymbol{C}$ *is false* and arrive to the conclusion that $\boldsymbol{H}$ cannot be true, i.e. $\overline{\boldsymbol{H}}$ is true; thus $\overline{\boldsymbol{C}}$ implies $\overline{\boldsymbol{H}}$. Since the contrapositive of $\boldsymbol{H} \Rightarrow \boldsymbol{C}$ is true, the direct statement is also true.

Contrapositive reasoning is often unconsciously used in our everyday life. For example, when looking on a shelf for a certain book with a green cover, one would skip (deny) all books of other colours (non-green).

Let us denote the statements:

$\boldsymbol{H}$: this is the required book; $\boldsymbol{C}$: this book has a green cover.

The respective negations are:

$\overline{\boldsymbol{H}}$: this is not the required book; $\overline{\boldsymbol{C}}$: the cover of this book is not green.

It is known that $\boldsymbol{H} \Rightarrow \boldsymbol{C}$ (if it is the needed book, then it is green). Then we start searching. The cover of the first book is blue (not green), hence it is not what we need: $\overline{\boldsymbol{C}} \Rightarrow \overline{\boldsymbol{H}}$; the second book is red (not green), so it is not the one we need: $\overline{\boldsymbol{C}} \Rightarrow \overline{\boldsymbol{H}}$; . . . the twenty seventh book is green, – let us look at the title, it may be the one we are looking for.

Let us notice that the converse of $\boldsymbol{H} \Rightarrow \boldsymbol{C}$ is not necessarily true, therefore the green book is not necessarily the one we need.

The properties of conditional statements are conveniently visualized by means of *Venn diagrams*, where statements are represented by points in the corresponding plane regions.

For our "territorial" example: *If I live in Edmonton, then I live in Alberta*, the diagram emerges naturally (see Figure 2.3.1).

A plane bounded domain represents Alberta, and everyone within this domain lives in Alberta. Edmonton is shown as a rectangle located approximately in the middle of the province. A person inside this rectangle illustrates the direct statement, the inhabitants of the Northern and Southern Alberta disprove, respectively, the converse and inverse, and a resident of the North West Territories supports the contrapositive statement.

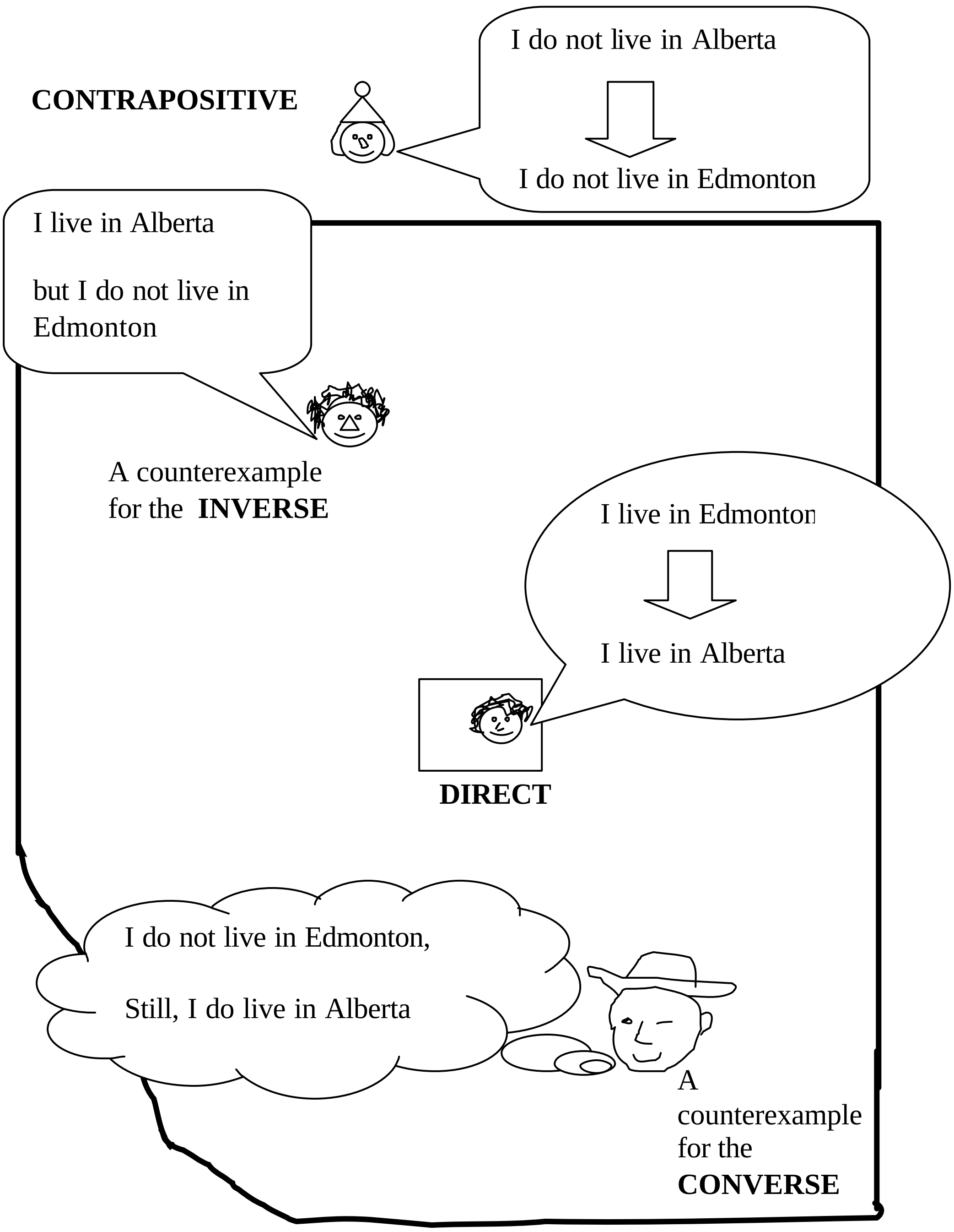

Figure 2.3.1

We shall conclude this section with the summary of logical relations between a direct and the related conditional statements.

Direct		*Contrapositive*
$H \Rightarrow C$	$\Leftrightarrow$	$\overline{C} \Rightarrow \overline{H}$

Converse		*Inverse*
$C \Rightarrow H$	$\Leftrightarrow$	$\overline{H} \Rightarrow \overline{C}$

Problems for sections 2.2 and 2.3

2.2-3 Theorems: the structure and proofs. Direct, converse, inverse, and contrapositive theorems.

1. In each of the following cases put a statement in the conditional form. Underline the hypothesis (condition) by a straight line and the conclusion – by a wavy line.
   a) Stop on red light.
   b) Wooden objects float.
   c) An isometry transforms a figure into a conruent figure.
   d) Two circles of equal radii are congruent .
   e) A statement based on empiric induction is not necessarily true.
   f) Axiom that follows from some other axioms of the same set, violates the independence property.
   g) A city located in Hungary is a European city.
   h) Decent people do not cheat.
   i) Two distinct lines have at most one common point.

2. Explain, what kind of proof is called direct. What kind of proof did you use in problems 12, 13, 42, 44, 52 of Section 1.3?

3. For each of the following statements in the left column choose its negation from the statements in the right column. Substantiate your choice based on properties of complements and diagrams.
   a) This animal is a dog. (i) This animal is a cat
   (ii) This is not an animal
   (iii)This animal is not a dog

b) The expression in brackets is positive.

(i) The expression in brackets is negative
(ii) The expression in brackets is either negative or zero.
(iii) The expression that stands outside the brackets is positive.

c) This number is divisible by 5.

(i) This number is prime
(ii) This number is divisible by any number except 5.
(iii) This number is not divisible by 5.

d) Segment a is congruent to segment b.

(i) Segment a is greater than segment b.
(ii) Segments a and b do not lie in the same line.
(iii) Either $a < b$ or $a > b$.

4. Propose an example of a statement and its negation. Propose also a few false (confusing) negations for your example.

5. Use RAA (proof by contradiction) to prove that two distinct lines may have at most one common point.

6. Use RAA to prove that in the same circle, or in congruent circles, equal chords subtend equal arcs.

7. For each of the following statements do the following: (i) put it in the conditional form; (ii) write its converse, inverse, and contrapositive statements (iii) determine if the converse is true; (iv) illustrate the statements by a Venn diagramm.
   a) All humans have one nose.
   b) Congruent circles have equal radii.
   c) An even number that is divisible by 3, is divisible by 6.
   d) A moving object that changes the direction of motion, is subjected to a force.

8. Make a true statement whose converse is false. Is the inverse false? Is the contrapositive false?

9. Make a statement whose converse is true. Reformulate the statement using the expressions if and only if, and conversely, and vice versa to show that the converse is also true. Is the inverse true? Is the contrapositive true?

10. For each of the following pairs of statements answer the following questions:
Is (a) necessary for (b)?
Is (a) sufficient for (b)?
Are (a) and (b) equivalent?
    (i) (a) Number k >2 is prime. (b) Number k is odd.
    (ii) (a) This is a mammal. (b) This is a whale.
    (i) (a) Each of these transformations is an isometry. (b) The composition of these transformations is an isometry.
    (ii) (a) $x^2 = 25$ (b) $x = 5$
    (iii) (a) This number is odd and divisible by 5. (b) The last digit of this natural number is 5.